NUMERICAL INTEGRATION OF THE EQUATION OF HEAT PROPAGATION FOR VARIABLE PHYSICAL CHARACTERISTICS

N. I. Nikitenko

Inzhenerno-Fizicheskii Zhurnal, Vol. 9, No. 4, pp. 512–516, 1965

UDC 536.2

A numerical method of calculating the temperature field in a fluid flow with variable physical characteristics is examined. The conditions of convergence and stability of the difference equation of heat propagation are investigated. As an example, certain results of calculations of a two-dimensional temperature field, performed on a digital computer, are presented.

Consider the transfer of heat in a fluid flow when the parameters λ, c, γ and the velocity w are functions of position, time, or temperature. A numerical method of integration of the differential equation of heat propagation

$$\frac{D(c\gamma t)}{d\tau} = \operatorname{div}(\lambda \operatorname{grad} t) \tag{1}$$

was described in [1, 2]. This paper is a continuation of the author's previous work. It examines the finite difference equation corresponding to the process of propagation of heat in a fluid flow with variable physical characteristics and investigates the conditions of convergence and stability. We transform (1) to the following form:

$$\frac{\partial t}{\partial \tau} = \frac{\lambda}{c\gamma}\sum_{k=1}^{3}\frac{\partial^2 t}{\partial x_k^2} + \sum_{k=1}^{3}\frac{\partial t}{\partial x_k}\left(\frac{1}{c\gamma}\frac{\partial\lambda}{\partial x_k} - w_k\right) - \frac{1}{c\gamma}\left[\frac{\partial(c\gamma)}{\partial\tau} + \sum_{k=1}^{3} w_k\frac{\partial(c\gamma)}{\partial x_k}\right]t. \tag{2}$$

To obtain the finite-difference approximation of (2), we represent the partial derivatives in the form of difference relations:

$$\frac{\partial t}{\partial \tau} = \frac{t_{\tau+l} - t}{l} + \varepsilon_\tau; \tag{3}$$

$$\frac{\partial \varphi}{\partial x_k} = \frac{\varphi_k'' - \varphi_k'}{2h_k} + \varepsilon'_{\varphi k};\ \varphi = t,\ \lambda,\ c\gamma; \tag{4}$$

$$\frac{\partial^2 t}{\partial x_k^2} = \frac{t_k'' + t_k' - 2t}{h_k^2} + \varepsilon_k''; \tag{5}$$

$$\frac{\partial (c\gamma)}{\partial \tau} = \frac{c\gamma - (c\gamma)_{\tau-l}}{l} + \varepsilon_{c\gamma}. \tag{6}$$

Here l and h_k are the space and time steps along the axes τ and x_k (k = 1, 2, 3), $t = t(x_1, x_2, x_3, \tau)$, $t_{\tau+1} = t(x_1, x_2, x_3, \tau + l) = t(\tau + l)$, $t_k = t(x_k + h_k)$, $t_k = t(x_k - h_k)$.

Values of approximation errors ε can be calculated from Taylor's interpolation formula. Their orders are respectively

$$\varepsilon_\tau = 0(l);\ \varepsilon'_{\varphi k} = 0(h_k^2);\ \varepsilon_k'' = 0(h_k^2);\ \varepsilon_{c\gamma} = 0(l). \tag{7}$$

Taking account of (3)–(6), we can write equation (2) in the following difference form:

$$u_{\tau+l} = \left(1 - 2\sum_{k=1}^{3} - M\right)u + \sum_{k=1}^{3} B_k[(1-c_k)u_k'' + (1+c_k)u_k'], \tag{8}$$

where

$$B_k = \frac{\lambda l}{c\gamma h_k^2},\ C_k = \frac{w_k h_k c\gamma}{2\lambda} - \frac{\lambda_k'' - \lambda_k'}{4\lambda},$$

$$M = \frac{l}{c\gamma}\left[\frac{c\gamma - (c\gamma)_{\tau-l}}{l} + \sum_{k=1}^{3} w_k\frac{(c\gamma)_k - (c\gamma)_k}{2h_k}\right].$$

The function $u_{\tau+l}$ approximates the real value of the temperature $t_{\tau+l}$ with error ($\xi = t - u$):

$$\xi_{\tau+l} = \left(1 - 2\sum_{k=1}^{3} B_k - M\right)\xi + \sum_{k=1}^{3} B_k[(1-C_k)\xi_k'' + (1+C_k)\xi_k'] + lR. \tag{9}$$

The function

$$R = -\varepsilon_\tau + \frac{\lambda}{c\gamma}\sum_{k=1}^{3}\varepsilon_k'' + \sum_{k=1}^{3}\varepsilon'_{tk}\left(\frac{1}{c\gamma}\frac{\partial\lambda}{\partial x_k} - w_k\right) + \frac{1}{c\gamma}\sum_{k=1}^{3}\varepsilon'_{\lambda k}\frac{\partial t}{\partial x_k} - \frac{t}{c\gamma}\left(\varepsilon_{c\gamma} + \sum_{k=1}^{3} w_k\varepsilon'_{c\gamma k}\right) \tag{10}$$

as follows from (7) is of the order of $0(h_k^2, l)$.

Proceeding to an investigation of the convergence of the difference equation (8) to the differential equation (2), we show that this is in fact the case if the conditions

$$2\sum_{k=1}^{3} B_k + M \leqslant 1 + pl,\ M \geqslant -pl, \tag{11}$$

$$|C_k| \leqslant 1 \tag{12}$$

are satisfied. The relations (11) establish that the order of l must be no less than h_k^2.

We denote by δ_τ a positive number which is less than than the greatest absolute value of ξ for all nodes of the space network at time τ. Then from (9) there follows the inequality

$$|\xi_{\tau+l}| < \left|1 - 2\sum_{k=1}^{3} B_k - M\right|\delta_\tau + \sum_{k=1}^{3} B_k\{|1-c_k| + |1+c_k|\}\delta_\tau + \beta l. \tag{13}$$

With account for (11) and (12), we rewrite (13) as follows:

$$|\xi_{\tau+l}| \le \delta_\tau (1-2pl) + \beta l,$$

from which there immediately results

$$|\xi_{ml}| \le \delta_0 (1+2pl)^m + [(1+2pl)^{m-1} + (1+2pl)^{m-2} + \dots + (1+2pl) + 1]\beta l \le \delta_0 (1+2pl)^m + m(1+2pl)^m \beta l. \quad (14)$$

If it is assumed that $\tau = ml$ is some bounded quantity, then

$$(1+2pl)^m \le l^{2p\tau}$$

also has some finite value. Since the quantity δ_0 can be calculated with any degree of accuracy [2], from inequality (14) it follows that

$$|\xi_\tau| \le Ch_k^2 \text{ and } \lim_{h\to 0} \xi_\tau \to 0, \quad (15)$$

where C is a positive number which does not depend on h_k.

Thus, given conditions (11), (12), the difference equation (8) permits one to calculate the temperature function with an error of order h_k^2. From (15) there directly follows the convergence of the difference equation (2).

In finding the necessary conditions of stability of the difference equation (8), we suppose that the distribution function of round-off errors η, which is given by the equation

$$\eta_{\tau+l} = \left(1-2\sum_{k=1}^{3} B_k - M\right)\eta + \sum_{k=1}^{3} B_k[(1-C_k)\eta_k'' + (1+C_k)\eta_k'], \quad (16)$$

can be expanded in series [3]

$$\eta_0 = \sum_{n_1}\sum_{n_2}\sum_{n_3} T_{n0} \exp\left(i\sum_{k=1}^{3}\mu_{nk}x_k\right). \quad (17)$$

Here

$$\mu_{nk} = \frac{n_k \pi}{N_k h_k};\ T_{n0} = (T_n)_{\tau=0};\ T_n = T_n(\tau, \mu_{n1}, \mu_{n2}, \mu_{n3}, h_1, h_2, h_3);$$

N_k is the number of steps h_k included in the projection of the region in question on the axis x_k; $n_k = 1, 2, \dots, N_k - 1$.

We represent the expression for the function η in the form of a series (17), in which T_{n0} is replaced by T_n. After substitution of η in (16) we immediately obtain

$$T_n = (\varphi_n)^{\tau/l}, \quad (18)$$

$$\varphi_n = 1-2\sum_{k=1}^{3} B_k - M + \sum_{k=1}^{3} B_k[(1-C_k)\exp(i\mu_{nk}h_k) + (1+C_k)\exp(-i\mu_{nk}h_k)],$$

or

$$\varphi_n = 1 - M - 2\sum_{k=1}^{3}(1-\cos\mu_{nk}h_k)B_k + i\sum_{k=1}^{3} 2B_kC_k\sin\mu_{nk}h_k$$

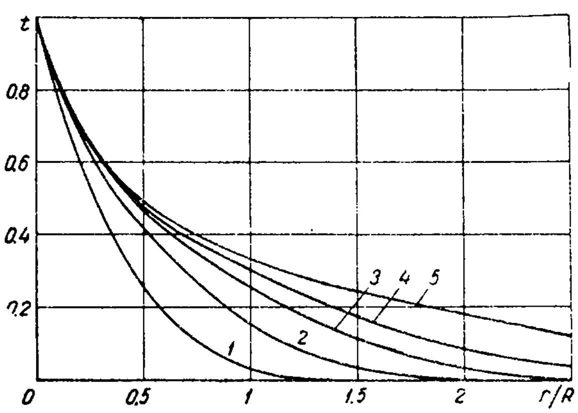
From (18) it follows that T_n will be bounded if

$$|\varphi_n|^2 \le 1 + Nl. \quad (19)$$

We set $C_k \le 1$ and denote by α the value of $\mu_{nk}h_k$, at which the function $|\varphi_n|$ is a maximum. Then the inequality (19) can be transformed to

$$(1-M)^2 - (1-\cos\alpha)\,2\left(\sum_{k=1}^{3} B_k\right)\left(1 - M - \sum_{k=1}^{3} B_k\right) \le 1 + Nl. \quad (20)$$

The inequality (20) holds if condition (11) is satisfied. Thus, the conditions of stability impose on the choice of space and time steps the same limitations as the conditions of convergence of the difference equation (18). They serve as a basis for constructing the space network and choosing the step l for numerical calculation of the temperature field in a fluid flow.



Nonsteady-state temperature field in a slot along the plane $y/y_0 = 0.75$: 1,2,3,4,5,) respectively, at times $\tau/3600 = 0.5$; 1; 1.5; 2; 3.52 sec. Curve 5 relates to conditions close to steady-state.

In conclusion, by way of example, we shall examine the problem of a nonsteady-state temperature field in a fluid flow moving in a slot-type channel. It is assumed that the motion is laminar, so that the velocity profile in the channel is parabolic:

$$w_x = 2w_0[1-(y/y_0)^2].$$

The following initial data are assumed: $t(x, y, 0) = 0$; $t(x, y_0, \tau) = t(x, -y_0, \tau) = 0$; $t(0, y, \tau) = 1$; $(\partial^2 t(b, y, \tau))/\partial x^2 = 0$ (b is the length of the channel); $B_x = B_y = 0.25$; $C_{max} = (2w_0h_xc\gamma)/2\lambda = 1$; $h_x/y_0 = h_y/h_0 = 0.25$; $l = 45$ sec; $w_0 = 1/3600$ m/sec.

The system consisting of (8) and the finite-difference approximations of the boundary conditions is solved by the pivot method at each moment of time $\tau = ml$. Successive transition from the layer ml to the layer $(m + 1)l$ determines the change of the temperature function in time. The figure presents the results of calculations performed on a digital computer.

As these calculations show, violation of conditions (11) and (12) usually makes it impossible to obtain the expected results.

NOTATION

p) positive constant; β) positive number which satisfies the condition $\beta \geq |R|$; δ_0) value of δ_τ at time $\tau = 0$; N) a certain constant; w_0) mean velocity; $2l_0$) width of channel.

REFERENCES

1. N. I. Nikitenko, IFZh, no. 8, 1963.
2. N. I. Nikitenko, Izv. vuzov, Aviatsionnaya tekhnika, no. 1, 1963.
3. J. Neumann and H. Goldstine, Bull. Am. Math. Soc., 53, 1947.

7 December 1964

Institute of Technical Thermophysics, Kiev